Note on the interpretation of two-dimensional theories of growing cavities

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It is shown in general how a two-dimensional flow can be justified as a physical approximation, notwithstanding the logarithmic singularity in pressure that occurs at infinity when the cavity expands or contracts at a varying rate. The argument presented, which affords a more natural interpretation than alternatives previously suggested, refers to the approximate equivalence—to a determinable degree of accuracy—between the hypothetical plane flow and the inner region of some real three-dimensional flow with small spanwise variations. The main ideas are illustrated by the example of a long ellipsoidal body which changes in volume while also undergoing shape perturbations.

1. Introduction

In the paper by Woods (1964) that precedes this note, a two-dimensional theoretical model is used to investigate the flow around a growing vapour- or air-filled cavity of finite section formed behind a hydrofoil in a stream of water. A similar model was also applied recently by Wang & Wu (1963) to the same problem, and its physical implications had previously been noted by Wu (1958). These authors have recognized that the infinite plane flows considered by them are basically unreal in that the pressure is unbounded at large distances from the cavity, and though reasonably convincing intuitive arguments have been put forward, the admissibility of this feature in a physical approximation does not appear to have been demonstrated definitely so far. The matter is taken up in this note.

The essential difficulty is easily explained as follows. If the rate of increase in the cavity volume (per unit span) is $Q(t)$ and the fluid is incompressible, then by continuity the mean velocity potential at a radius ϖ extending beyond the cavity is $\Phi = (Q/2\pi)\log\varpi$. (The possible presence of circulation is ignored here, but otherwise the argument would need only trivial modification and the conclusion would be unaffected.) Hence, in consequence of its dependence on $-\partial\Phi/\partial t$, the pressure is unbounded for $\varpi \to \infty$ if Q varies and the cavity pressure is finite. More meaningfully, one may say that indefinitely large pressures are necessary at great distances in order to generate a volume change in the cavity. As was pointed out by Wang & Wu (1963), however, the extrapolation of the plane flow to infinite distances is merely a simplifying idealization, and any real flow is necessarily three-dimensional 'in the large'; thus the pressure singularity is avoided by the flow becoming three-dimensional at finite distances.

This peculiarity of plane flows was illustrated by Birkhoff (1950, p. 34) with reference to the cavity produced by a 'two-dimensional underwater explosion'. If the water is supposed incompressible and the pressures finite, no expansion of the cavity appears to be possible. As a resolution of the paradox Birkhoff suggested that 'at very large distances the effect of compressibility in absorbing the expansion is dominant'; but the present view of the matter radically contradicts this interpretation, which may be criticized for appealing to an extraneous physical property as a way out of a purely theoretical difficulty. No practical situation seems conceivable in which, at large distances from an approximately two-dimensional cavity, the impediment to expansion is relieved by compression of the water rather than by the more obviously providential means of a three-dimensional flow. [Note that the relative importance of these alternative mechanisms in a practical example would be indicated by a comparison between the length scale L of spanwise variations (*vide infra*) and the wavelength of sound waves in water; only if the former exceeded the latter could one presume compressibility to be the predominant factor. But even for vibrations at a frequency of 100 c/s, say, which is a rather high value for experimental cavities, the wavelength is more than 48 ft and so it seems extremely unlikely that L could be other than a great deal shorter!] Several other theoretical devices, more or less artificial, have also been proposed for overcoming the difficulty in question; for example, Geurst (1961) listed four possible ways in all, although unfortunately he missed the interpretation that now appears to be the only really meaningful one.

According to the view to be developed here, which consolidates the ideas proposed by Wang & Wu (1963) in particular, the proper rationale for an infinite two-dimensional model is to recognize as an essential feature that the cavity volume changes are 'driven' by infinite pressures or suctions at $\varpi = \infty$, and to regard $Q(t)$ as an *arbitrary* property of the hypothetical system. Thus the system is obviously unrealistic taken *in toto*, but still it provides a self-consistent model within whose framework the dependence on $Q(t)$ of properties such as cavity length may be calculated. To justify the theory as a physical approximation, one has then to prove that such a dependence on $Q(t)$ is nearly the same as in a real system where, at large ϖ, the outflow accompanying cavity expansion is accommodated by an essentially three-dimensional displacement of fluid. This is the main object of this note, namely to substantiate analytically the intuitive arguments given by Wang & Wu.

The point of interpretation made outstanding by the present considerations is that in practice Q will necessarily depend on factors outside the scope of any two-dimensional theory, so that in developing such a theory it is pointless to regard Q other than as an independent property or to attempt to allow empirically for effects such as evaporation and entrainment on which Q ultimately depends. To deduce Q *a priori* for any real system a three-dimensional theory is essential, in which of course Q would have to be represented as a function of spanwise position.

It may be remarked, incidentally, that a rather similar difficulty is presented by the need to represent changes in the circulation about the hydrofoil-cavity combination. But the device of a trailing vortex sheet used by Woods (1964) allows for this factor satisfactorily within the framework of the two-dimensional model, and so there is no necessity to refer to the spanwise stretching of 'bound' vortex lines, which could provide an alternative rationale analogous to that for the variations in cavity volume. For simplicity, therefore, circulation will be ignored in the following discussion; that is, the two-dimensional velocity potential is assumed to be single-valued.

2. Analytical discussion

The crux of the matter is to match two-dimensional potentials to three-dimensional ones. More precisely, we need to show how, in an approximately two-dimensional situation (i.e. with small spanwise variations), the solution to the strictly two-dimensional problem posed in respect of a typical cross-section may approximate closely to a certain 'infield' yet may be linked smoothly to a three-dimensional solution for the 'outfield', where the radial distance from the cavity is comparable with the length scale of the spanwise variations. The analytical means for thus matching a two-dimensional source flow is recognizable at once in the fact that the velocity potential† in the outfield may be synthesized from terms like

$$\phi = \cos \alpha z K_0(\alpha \varpi),$$

where z is the spanwise co-ordinate. Here α^{-1} is large in comparison with the dimensions of the hydrofoil-cavity section, but $\alpha\varpi$ is $O(1)$ or greater. This potential has the property $\phi \to 0$ for $\varpi \to \infty$ and so meets the physical requirement that the pressure is bounded at infinity; but since the Bessel function has logarithmic behaviour near the origin, it can be identified for $\alpha\varpi$ small with the two-dimensional source potential $\Phi = \log \varpi$ applicable within the infield.

The point may be argued more formally as follows. Consider a three-dimensional situation where the representative length scale of the hydrofoil-cavity cross-section is l and that of the spanwise variations, over a certain central region excluding the lateral extremities, is $L\,(\gg l)$. Let x and y be the (dimensional) co-ordinates in cross-sectional planes and again z the spanwise co-ordinate.

Then, in terms of the dimensionless co-ordinates $X = x/l$, $Y = y/l$ and $Z = z/L$, Laplace's equation for the velocity potential takes the form

$$\frac{\partial^2\phi}{\partial X^2} + \frac{\partial^2\phi}{\partial Y^2} + \epsilon^2\frac{\partial^2\phi}{\partial Z^2} = 0, \tag{1}$$

in which $\epsilon^2 = (l/L)^2$ is a very small number by hypothesis; and clearly each of the three second derivatives in (1) is at most $O(1)$ in the central region. The infield may be defined as that part of this region over which $R = (X^2 + Y^2)^{\frac{1}{2}}$ is small in comparison with ϵ^{-1}. But since the system is unbounded in R it must also include an outfield where $\epsilon R \geqslant O(1)$, however small ϵ may be, and here all three terms in (1) become of comparable magnitude. Consequently there can be no approximate solution developed in powers of ϵ that is uniformly valid for all R.

† To save unnecessary writing, only that part of the velocity potential due to the presence of the hydrofoil and cavity will be considered explicitly; that is, the component representing the undisturbed stream, Ux say, will be left implicit everywhere.

For the infield, however, one can derive an approximate solution to, in principle, an arbitrary degree of accuracy according to the scheme

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \dots, \tag{2}$$

provided that at the outskirts of this region it can be matched to an admissible solution for the outfield.

In the first approximation, (1) gives

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)\phi_0 = 0,$$

and so we may write

$$\phi_0(X, Y, Z) = f(Z)\,\Phi(X, Y), \tag{3}$$

where $\Phi(X, Y)$ is a conjugate function. The interior boundary condition will be independent of Z-derivatives in this approximation, and thus a two-dimensional problem such as considered by Woods is posed, the only specification still lacking being the behaviour of Φ for large R. [The exact forms of the boundary conditions will contain terms which are $O(\epsilon)$, but (1) shows that $\phi_0 + \epsilon\phi_1$ is also expressible in the form (3). Hence the second approximation can be derived again using this form of velocity potential. This would be a 'slender-body approximation' in the general sense of the term.]

At this point it becomes necessary to specify the overall physical problem rather more precisely. We must in fact recognize that the rate of expansion of the cavity cross-section, $Q(Z, t)$, is dependent on the complete three-dimensional flow, but that this together with the cavity pressure are the *only* variable characteristics of the approximately two-dimensional flow in a plane z = const. that are not determined to the first approximation by the local behaviour of the cavity. In other words, there is no incidental factor affecting the outfield in the 'central region', such as the presence of additional boundaries at distances $O(\epsilon^{-1})$. This means that the flow perturbation in the outfield will be very nearly axially symmetric, being primarily due to the cavity expansion, although of course some residual effect of the asymmetric flow in the vicinity of the cavity will inevitably extend to the outfield. The important point is that the asymmetric part of the flow must diminish with radial distance even within the infield, since it is determined by the local state of the cavity and not driven from outside the infield.

Hence, since Φ is assumed to be single-valued, it may be expressed in terms of R and $\omega = \tan^{-1}(Y/X)$ in the general form

$$\Phi = A + a_0 \log R + \sum_{n=1}^{\infty} \frac{a_n}{R^n} \cos(n\omega + \delta_n), \tag{4}$$

which is complete outside a circle R = const. enclosing the hydrofoil-cavity cross-section. Here A, the a's and the δ's are functions of Z alone, and potentials of the type $R^n \cos(n\omega + \delta_n)$ have been excluded for the reason explained in the preceding paragraph. If ϵ is sufficiently small so that the infield extends to fairly large R, then the terms involving R^{-n} become very small in the 'overlapping region' where the infield merges into the outfield and matching is to be accomplished;

hence there is little significance in identifying the corresponding components of the outfield, but since this can be done very simply we shall briefly point out the procedure below. The more important task, of course, is to match the two leading terms in (4), identifying them with the solution of the implicit three-dimensional problem.

For the outfield the general solution of (1) having the property $\phi \to 0$ for $R \to \infty$ may be expressed in the following form, where the Z-variation is represented by Fourier integrals:

$$\phi = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} e^{i\epsilon kZ} \cos[n\omega + \eta_n(k)]\, K_n(\epsilon kR)\, g_n(k)\, dk. \tag{5}$$

Alternatively the solution might be expressed in terms of Fourier series over a finite range of Z (e.g. the breadth of the 'central region'), but the essential argument given as follows would be unaffected. From the definition of Z it is clear that $g_n(k)$, or the coefficients of the Fourier series, will become negligibly small for $k > O(1)$.

The symmetric component of (5) is, with $\eta_0 = 0$ taken for simplicity,

$$\overline{\phi} = \int_{-\infty}^{\infty} e^{i\epsilon kZ} K_0(\epsilon kR)\, g_0(k)\, dk, \tag{6}$$

and we need to match this to the source potential for the infield. In the 'overlapping region' we have $\epsilon R \ll 1$, and since only $k = O(1)$ is relevant we have

$$K_0(\epsilon kR) = -\{\log(\tfrac{1}{2}\epsilon kR) + \gamma\} + O(\epsilon^2 R^2).$$

Putting $\log(\frac{1}{2}\epsilon kR) = \log R + \log(\frac{1}{2}\epsilon k)$, we see that the coefficient of $\log R$ and the remainder in the first approximation to (6) are both functions of Z alone. Thus the required matching with $\overline{\phi}_0 = A(Z) + a_0(Z)\log R$ is precisely achieved.

Since if $n \geqslant 1$ we have, for ϵkR small,

$$K_n(\epsilon kR) \sim \tfrac{1}{2}(n-1)!\,(\tfrac{1}{2}\epsilon kR)^{-n},$$

the remaining terms in (4) may similarly be matched to (5). But we observe that $g_n = O(\epsilon^n)$ in this case, which confirms our previous remarks as to the smallness of the asymmetric part of the outfield in the present approximation.

3. Example

To illustrate the foregoing ideas the flow caused by the uniform expansion or contraction of a long ellipsoidal body will be examined. This example is the simplest available whereby the general interpretation of the 'infield' can be demonstrated explicitly.

Consider the ellipsoid of revolution

$$x^2 + y^2 + \epsilon^2 z^2 = a^2, \tag{7}$$

with equatorial diameter $2a$ and polar diameter $2a/\epsilon$, $= 2b$ say. If a varies with time yet ϵ is constant, the surface remains similar to itself while the volume changes at a rate $dV/dt = 4\pi\epsilon^{-1}a^2\dot{a}$. It is supposed that the body expands or contracts in this basic form but also suffers perturbations from it which leave the volume unaffected. The body is surrounded by fluid which is motionless at infinity and whose flow at finite distances is irrotational, having been started from rest.

The first aim is to show that, if $\epsilon \ll 1$, there exists an infield throughout which the primary flow approximates to a two-dimensional source flow in each cross-sectional plane. The second is to show that the set of three-dimensional harmonics which describes the perturbed motion is equivalent to a set of two-dimensional ones within the infield.

We take prolate spheroidal co-ordinates (ρ, θ, ω) such that

$$\left.\begin{aligned} z = \rho\cos\theta, \quad \varpi = (x^2+y^2)^{\frac{1}{2}} = (\rho^2-c^2)^{\frac{1}{2}}\sin\theta, \\ x = \varpi\cos\omega, \quad y = \varpi\sin\omega. \end{aligned}\right\} \tag{8}$$

The surfaces $\rho =$ const. and $\theta =$ const. are confocal ellipsoids and hyperboloids of two sheets, respectively, with their common foci at $(0,\, 0,\, \pm c)$. In particular, the boundary (7) is given by $\rho = b$, $(\rho^2-c^2)^{\frac{1}{2}} = a$.

The velocity potential for the primary fluid motion is

$$\phi = -\frac{a^2\dot{a}}{\epsilon c}Q_0\left(\frac{\rho}{c}\right) = -\frac{a^2\dot{a}}{2\epsilon c}\log\frac{\rho+c}{\rho-c} \tag{9}$$

(cf. Lamb 1932, p. 150). That (9) satisfies the kinematical boundary conditions at the surface of the ellipsoid may be readily verified, and we observe that at great distances from the origin, such that $\rho \gg c$ and so $\rho \to r = (x^2+y^2+z^2)^{\frac{1}{2}}$, (9) gives

$$u_r = \frac{\partial\phi}{\partial r} \to \frac{a^2\dot{a}}{\epsilon r^2}. \tag{10}$$

Hence the total flow outwards is given correctly as $4\pi\epsilon^{-1}a^2\dot{a}$, equalling dV/dt as previously expressed.

Now, the infield around the ellipsoid may be specified by

$$\varpi/c\sin\theta = O(\lambda), \tag{11}$$

where λ is a small fraction such that λ^2 is negligible in the over-all approximation to be accepted. More simply, if the vicinity of the poles is excluded according to the criterion that $\sin\theta$ is not small, the definition may be taken as just

$$\varpi/c = O(\lambda), \tag{12}$$

which is equivalent to $\varpi/a = O(\lambda/\epsilon)$, since $c \doteqdot a/\epsilon$; thus the lateral extent of the infield is large compared with the cross-section of the ellipsoid when $\lambda/\epsilon \gg 1$, which is the general case in view.

By (8) we have

$$\rho = c\left\{1+\frac{\varpi^2}{2c^2\sin^2\theta}+O(\lambda^4)\right\}, \tag{13}$$

and

$$\theta = \cos^{-1}\left[\frac{z}{c}\{1+O(\lambda^2)\}\right]. \tag{14}$$

Substituting (13) in (9), we obtain after reduction

$$\phi = a\dot{a}\left\{\log\left(\frac{\varpi}{2c\sin\theta}\right)+O(\lambda^2)\right\}. \tag{15}$$

But (14) shows θ to be a function of z alone if $O(\lambda^2)$ is negligible. Hence, over the infield, the primary radial velocity in any plane z = const. is approximately

$$u_\varpi = \frac{\partial\phi}{\partial\varpi} = \frac{a\dot{a}}{\varpi} = \frac{\varpi_0\dot{\varpi}_0}{\varpi}, \tag{16}$$

where ϖ_0 is the radius of the section of the ellipsoid at the given z. Thus, as was to be shown, the flow approximates to the flow in two dimensions due to the expansion of a circular boundary of radius ϖ_0.

It remains to demonstrate a similar property for perturbations from this basic flow. There is no need to consider any particular boundary-value problem for the deformed ellipsoid, however, since the present object will be met merely by showing that the general solid-harmonic component of an arbitrary perturbation has a quasi two-dimensional representation within the infield. But it is implied, of course, that the deformation of the ellipsoid has significant z-variations only on a scale comparable with the length of the infield, or at least considerably larger than the equatorial diameter.

We first consider the symmetric part of the perturbed flow, for which the velocity potential may be expressed in the form

$$\phi = \sum_{s=1}^{\infty} A_s Q_s(\rho/c)\, P_s(\cos\theta). \tag{17}$$

But we have in general, for $\zeta > 1$,

$$Q_s(\zeta) = \tfrac{1}{2}\zeta^{-s}\left\{\log\frac{\zeta+1}{\zeta-1} + O(1)\right\}$$

(cf. Jeffreys & Jeffreys 1946, § 24.17). Hence, when the approximations (13) and (14) are substituted, it appears directly that over the infield the expression (17) is equivalent to

$$\phi = f_0(z)\log\varpi + g_0(z) \tag{18}$$

if $O(\lambda^2)$ is neglected. Hence the effect in each cross-sectional plane is simply an augmentation or diminution of the two-dimensional source flow represented by (16), and clearly $f_0(z)$ will be determined by the condition that the net flow out of a circle ϖ = const. equals the expansion rate of the deformed cross-section of the ellipsoid.

The part of the disturbance dependent on ω can be represented by

$$\phi = \sum_{m=0}^{\infty}\sum_{n=1}^{m} Q_m^n(\rho/c)\, P_m^n(\cos\theta)\,[A_{mn}\cos n\omega + B_{mn}\sin n\omega]. \tag{19}$$

But we have that, near $\zeta = 1$,

$$Q_m^n(\zeta) \sim \text{const.} \times (\zeta^2-1)^{-\frac{1}{2}n},$$

the fractional error in the expression being $O(\zeta-1)$ for $n > 1$ but

$$O\{(\zeta-1)\log(\zeta-1)\} \quad \text{for} \quad n = 1$$

(cf. Jeffreys & Jeffreys 1946). Hence, when the approximations (13) and (14) are used, and both $O(\lambda^2)$ and $O(\lambda^2 \log \lambda)$ are neglected, an equivalent form of (19) for the infield is seen to be

$$\phi = \sum_{n=1}^{\infty} \frac{1}{\varpi^n} \{f_n(z) \cos n\omega + h_n(z) \sin n\omega\}. \tag{20}$$

Thus, as we needed to show, in each cross-sectional plane the asymmetric part of the velocity potential is composed from the set of two-dimensional harmonics $\varpi^{-n}(\cos n\omega, \sin n\omega)$.

4. Conclusion

It has been shown how, in the study of cavity growth behind hydrofoils, a two-dimensional theory such as Woods (1964) has used can be justified as a physical approximation when spanwise variations are small on the scale of the cavity cross-section. Whereas an unbounded pressure at infinity can be allowed as an essential feature of the hypothetical plane flow that is considered, and indeed it must be allowed in order to rationalize the theory in the simplest way, the unreality of the model in this regard is immaterial to its applicability over a limited region (the 'infield') of an actual flow. But the practical usefulness of this type of theory is necessarily restricted by the fact that $Q(t)$, the rate of expansion of the cavity section, must be taken as an independent property since in reality it is inevitably dependent on three-dimensional effects.

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